

EXPANSION SERIES OF $f(x) = x^x$ AND CHARACTERIZATION OF ITS COEFFICIENTS

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ABSTRACT. In this paper we study the development in Taylor series of the function $f(x) = x^x$. Section 1 establishes a recursive relationship between successive derivatives of f by using the coefficients $\Omega(\cdot, \cdot)$ defined therein. From recursion between the derivatives you get one general description of them (section 2). Finally, section 3 has the main result, the expansion series of f . Section 4 deals with numbers $\Omega(\cdot, \cdot)$: characterization (4.1), its relationship with rencontre numbers (4.2) and its emergence as coefficients in certain polynomial families. The specific use of some of these polynomials allows eventually go deeper in the description of the series.

1. RECURSION OF THE DERIVATIVES OF $f(x) = x^x$

For the series expansion of the function one needs a description of its derivatives, but $f'(x)$ is not immediately determined. Instead of deriving f , f' can be found deriving $\ln(f(x))$.

On one hand,

$$\frac{\partial}{\partial x} \ln(f(x)) = \frac{f'(x)}{f(x)}$$

on the other hand,

$$\ln(f(x)) = \ln(x^x) = x \ln(x)$$

and therefore

$$\frac{\partial}{\partial x} \ln(f(x)) = \ln(x) + 1$$

Joining both equalities one gets

$$(1.1) \quad f'(x) = f(x) \ln(x) + f(x)$$

Next derivatives can be found deriving successively f' . Note that f appears in the expression of f' . This fact provides a recursive relationship between the derivatives of f , relationship characterized from the numbers $\Omega(\cdot, \cdot)$ defined below:

Definition 1.1. Given $a \in \mathbb{Z}_{>0}$ and $b \in \mathbb{Z}_{\geq 0}$, $\Omega(a, b)$ is defined by

$$\begin{cases} \Omega(a, 0) = 1 \\ \Omega(a, 1) = \Omega(a - 1, 1) + 1 \\ \Omega(a, b) = \Omega(a - 1, b) - (b - 1)\Omega(a - 1, b - 1) \text{ if } b = 2, \dots, a - 1 \\ \Omega(a, b) = 0 \text{ if } b > a \end{cases}$$

Numbers $\Omega(\cdot, \cdot)$ allow to relate the derivatives of f , as shown in the following proposition:

Proposition 1.2. Let $f^{(n)}(x)$ be the n -th derivative of $f(x) = x^x$, then

$$(1.2) \quad f^{(n)}(x) = f^{(n-1)}(x) \ln(x) + \sum_{i=0}^{n-1} \Omega(n, i) \frac{f^{(n-1-i)}(x)}{x^i}$$

Proof. By induction. If $n = 1$, then

$$f'(x) = f(x) \ln(x) + \Omega(1, 0) \frac{f(x)}{x^0} = f(x) \ln(x) + f(x)$$

which is right (1.1). Suppose now that the equation (1.2) is right for n and derive it:

$$\begin{aligned} f^{(n+1)}(x) &= f^{(n)}(x) \ln(x) + \frac{f^{(n-1)}(x)}{x} \\ &+ \sum_{i=0}^{n-1} \Omega(n, i) \frac{f^{(n-i)}(x)x^i - f^{(n-1-i)}(x)ix^{i-1}}{x^{2i}} \end{aligned}$$

The summation can be simplified separating it into two parts and resetting indexes in one of them:

$$\begin{aligned} &\sum_{i=0}^{n-1} \Omega(n, i) \frac{f^{(n-i)}(x)x^i - f^{(n-1-i)}(x)ix^{i-1}}{x^{2i}} \\ &= \sum_{i=0}^{n-1} \Omega(n, i) \frac{f^{(n-i)}(x)}{x^i} - \sum_{i=0}^{n-1} \Omega(n, i) \frac{f^{(n-1-i)}(x)i}{x^{i+1}} \\ &= \sum_{i=0}^{n-1} \Omega(n, i) \frac{f^{(n-i)}(x)}{x^i} - \sum_{i=1}^n \Omega(n, i-1) \frac{f^{(n-i)}(x)(i-1)}{x^i} \\ &= \Omega(n, 0) f^{(n)}(x) + \sum_{i=1}^{n-1} \Omega(n, i) \frac{f^{(n-i)}(x)}{x^i} \\ &- \Omega(n, n-1) \frac{f(x)(n-1)}{x^n} - \sum_{i=1}^{n-1} \Omega(n, i-1) \frac{f^{(n-i)}(x)(i-1)}{x^i} \\ &= \Omega(n, 0) f^{(n)}(x) - \Omega(n, n-1) \frac{f(x)(n-1)}{x^n} \\ &+ \sum_{i=1}^{n-1} \frac{f^{(n-i)}(x)}{x^i} (\Omega(n, i) - (i-1)\Omega(n, i-1)) \end{aligned}$$

Now we can apply the definition of $\Omega(\cdot, \cdot)$ to write the last summation in a more appropriate way:

$$\begin{aligned} &\sum_{i=1}^{n-1} \frac{f^{(n-i)}(x)}{x^i} (\Omega(n, i) - (i-1)\Omega(n, i-1)) \\ &= \frac{f^{(n-1)}(x)}{x} \Omega(n, 1) + \sum_{i=2}^{n-1} \frac{f^{(n-i)}(x)}{x^i} (\Omega(n, i) - (i-1)\Omega(n, i-1)) \\ &= \frac{f^{(n-1)}(x)}{x} \Omega(n, 1) + \sum_{i=2}^{n-1} \frac{f^{(n-i)}(x)}{x^i} \Omega(n+1, i) \end{aligned}$$

and replacing in the expression of $f^{(n+1)}(x)$, we have:

$$\begin{aligned}
f^{(n+1)}(x) &= f^{(n)}(x) \ln(x) + \frac{f^{(n-1)}(x)}{x} + \Omega(n, 0)f^{(n)}(x) \\
&\quad - \Omega(n, n-1) \frac{f(x)(n-1)}{x^n} + \frac{f^{(n-1)}(x)}{x} \Omega(n, 1) \\
&\quad + \sum_{i=2}^{n-1} \frac{f^{(n-i)}(x)}{x^i} \Omega(n+1, i) \\
f^{(n)}(x) \ln(x) &+ \frac{f^{(n-1)}(x)}{x} (\Omega(n, 1) + 1) + \Omega(n, 0)f^{(n)}(x) \\
&\quad - \Omega(n, n-1) \frac{f(x)(n-1)}{x^n} + \sum_{i=2}^{n-1} \frac{f^{(n-i)}(x)}{x^i} \Omega(n+1, i)
\end{aligned}$$

Finally, by definition of $\Omega(\cdot, \cdot)$, we know that $\Omega(n, 1) + 1 = \Omega(n+1, 1)$, $\Omega(n, 0) = \Omega(n+1, 0)$ and $-\Omega(n, n-1)(n-1) = \Omega(n+1, n)$ so

$$\begin{aligned}
f^{(n+1)}(x) &= f^{(n)}(x) \ln(x) + \frac{f^{(n-1)}(x)}{x} \Omega(n+1, 1) + \Omega(n+1, 0)f^{(n)}(x) \\
&\quad + \Omega(n+1, n) \frac{f(x)}{x^n} + \sum_{i=2}^{n-1} \frac{f^{(n-i)}(x)}{x^i} \Omega(n+1, i) \\
&= f^{(n)}(x) \ln(x) + \sum_{i=0}^n \frac{f^{(n-i)}(x)}{x^i} \Omega(n+1, i)
\end{aligned}$$

□

2. GENERAL EXPRESSION OF THE DERIVATIVES OF x^x

In section 1 we have obtained a recursive relationship of the derivatives of x^x . Through this recursion and the next definition, a general direct (non-recursive) expression of the derivatives can be found (proposición 2.2):

Definition 2.1. For all $n, k \in \mathbb{Z}_{\geq 0}$ we define

$$\begin{cases} \Delta_0^{(n)}(x) = 1 \\ \Delta_k^{(n)}(x) = \Delta_k^{(n-1)}(x) + \sum_{i=0}^{k-1} \Omega(n, i) \frac{1}{x^i} \Delta_{k-1-i}^{(n-1-i)}(x) \text{ if } k = 1, \dots, n \\ \Delta_k^{(n)}(x) = 0 \text{ if } k > n \end{cases}$$

Proposition 2.2. Let $f^{(n)}(x)$ be the n -th derivative of the function $f(x) = x^x$. Then

$$(2.1) \quad f^{(n)}(x) = f(x) \sum_{i=0}^n \Delta_i^{(n)}(x) \ln^{n-i}(x)$$

Proof. As in the previous proposition, by induction. If $n = 1$, according to (2.1) we get

$$f'(x) = f(x) \left(\Delta_0^{(1)}(x) \ln(x) + \Delta_1^{(1)}(x) \right)$$

which is right due to $\Delta_0^{(1)}(x) = \Delta_1^{(1)}(x) = 1$.

Let's suppose now that (2.1) is right for $1, \dots, n$. From proposition 1.2 we know

$$\begin{aligned} f^{(n+1)}(x) &= f^{(n)}(x) \ln(x) + \sum_{i=0}^n \Omega(n+1, i) \frac{1}{x^i} f^{(n-i)}(x) \\ &= \underbrace{f^{(n)}(x) [\ln(x) + \Omega(n+1, 0)]}_A + \underbrace{\sum_{i=1}^n \Omega(n+1, i) \frac{1}{x^i} f^{(n-i)}(x)}_B \end{aligned}$$

On one hand, because of the induction hypothesis,

$$\begin{aligned} A &= [\ln(x) + \Omega(n+1, 0)] f(x) \sum_{j=0}^n \Delta_j^{(n)}(x) \ln^{n-j}(x) \\ &= f(x) \left[\sum_{j=0}^n \Delta_j^{(n)}(x) \ln^{n+1-j}(x) + \sum_{j=0}^n \Delta_j^{(n)}(x) \ln^{n-j}(x) \Omega(n+1, 0) \right] \\ &= f(x) \left[\Delta_0^{(n)}(x) \ln^{n+1}(x) + \Delta_n^{(n)}(x) \Omega(n+1, 0) \right. \\ &\quad \left. + \sum_{j=1}^n \Delta_j^{(n)}(x) \ln^{n+1-j}(x) + \sum_{j=0}^{n-1} \Delta_j^{(n)}(x) \ln^{n-j}(x) \Omega(n+1, 0) \right] \\ &= f(x) \left[\Delta_0^{(n)}(x) \ln^{n+1}(x) + \Delta_n^{(n)}(x) \Omega(n+1, 0) \right. \\ &\quad \left. + \sum_{j=0}^{n-1} \Delta_{j+1}^{(n)}(x) \ln^{n-j}(x) + \sum_{j=0}^{n-1} \Delta_j^{(n)}(x) \ln^{n-j}(x) \Omega(n+1, 0) \right] \\ &= f(x) \left[\Delta_0^{(n)}(x) \ln^{n+1}(x) + \Delta_n^{(n)}(x) \Omega(n+1, 0) \right. \\ &\quad \left. + \sum_{j=0}^{n-1} \left(\Delta_{j+1}^{(n)}(x) + \Omega(n+1, 0) \Delta_j^{(n)}(x) \right) \ln^{n-j}(x) \right] \end{aligned}$$

On the other hand,

$$\begin{aligned} B &= \sum_{i=1}^n \Omega(n+1, i) \frac{1}{x^i} \sum_{j=0}^{n-i} \Delta_j^{(n-i)}(x) f(x) \ln^{n-i-j}(x) \\ &= \sum_{i=1}^n \sum_{j=0}^{n-i} \Omega(n+1, i) \frac{1}{x^i} \Delta_j^{(n-i)}(x) f(x) \ln^{n-i-j}(x) \end{aligned}$$

now, rearranging the summands of B according to their degree g in $\ln^g(x)$,

$$B = \sum_{g=0}^{n-1} \sum_{h=1}^{n-g} \Omega(n+1, h) \frac{1}{x^h} \Delta_{n-g-h}^{(n-h)}(x) f(x) \ln^g(x)$$

With these arrangements, we can sum again A and B :

$$\begin{aligned} A + B &= \Delta_0^{(n)}(x) f(x) \ln^{n+1}(x) + \Omega(n+1, 0) \Delta_n^{(n)}(x) f(x) \\ &\quad + \sum_{j=0}^{n-1} \left[\Delta_{j+1}^{(n)}(x) + \Omega(n+1, 0) \Delta_j^{(n)}(x) \right] f(x) \ln^{n-j}(x) \end{aligned}$$

$$\begin{aligned}
& + \sum_{g=0}^{n-1} f(x) \ln^g(x) \sum_{h=1}^{n-g} \Omega(n+1, h) \frac{1}{x^h} \Delta_{n-g-h}^{(n-h)}(x) \\
& = \Delta_0^{(n)}(x) f(x) \ln^{n+1}(x) + f(x) \sum_{h=0}^n \Omega(n+1, h) \frac{1}{x^h} \Delta_{n-h}^{(n-h)}(x) \\
& \quad + \sum_{g=1}^{n-1} f(x) \ln^g(x) \sum_{h=1}^{n-g} \Omega(n+1, h) \frac{1}{x^h} \Delta_{n-g-h}^{(n-h)}(x) \\
& \quad + \sum_{j=0}^{n-1} \left[\Delta_{j+1}^{(n)}(x) + \Omega(n+1, 0) \Delta_j^{(n)}(x) \right] f(x) \ln^{n-j}(x)
\end{aligned}$$

With the change $n-j = g$, $A+B$ can be written as follows

$$\begin{aligned}
A+B & = \Delta_0^{(n)}(x) f(x) \ln^{n+1}(x) + f(x) \sum_{h=0}^n \Omega(n+1, h) \frac{1}{x^h} \Delta_{n-h}^{(n-h)}(x) \\
& \quad + \sum_{j=1}^{n-1} \left[\Delta_{j+1}^{(n)}(x) + \sum_{h=0}^j \Omega(n+1, h) \frac{1}{x^h} \Delta_{j-h}^{(n-h)}(x) \right] f(x) \ln^{n-j}(x) \\
& \quad + \left[\Delta_1^{(n)}(x) + \Omega(n+1, 0) \Delta_0^{(n)}(x) \right] f(x) \ln^n(x)
\end{aligned}$$

which rearranged is

$$\begin{aligned}
A+B & = \Delta_0^{(n)}(x) f(x) \ln^{n+1}(x) + \left[\Delta_1^{(n)}(x) + \Omega(n+1, 0) \Delta_0^{(n)}(x) \right] f(x) \ln^n(x) + \\
& \quad \sum_{j=1}^{n-1} \left[\Delta_{j+1}^{(n)}(x) + \sum_{h=0}^j \Omega(n+1, h) \frac{1}{x^h} \Delta_{j-h}^{(n-h)}(x) \right] f(x) \ln^{n-j}(x) + \\
& \quad f(x) \left[\sum_{h=0}^n \Omega(n+1, h) \frac{1}{x^h} \Delta_{n-h}^{(n-h)}(x) \right]
\end{aligned}$$

Finally, due to

$$\Delta_i^{(n+1)}(x) = \Delta_i^{(n)}(x) + \sum_{h=0}^{i-1} \Omega(n+1, h) \frac{1}{x^h} \Delta_{i-1-h}^{(n-h)}(x)$$

and

$$\Delta_0^{(n+1)}(x) = \Delta_0^{(n)}(x)$$

the adjustment of the expression can be completed:

$$\begin{aligned}
A+B & = \Delta_0^{(n+1)}(x) f(x) \ln^{n+1}(x) + \Delta_1^{(n+1)}(x) f(x) \ln^n(x) \\
& \quad + \sum_{j=1}^{n-1} \Delta_{j+1}^{(n+1)}(x) \ln^{n-j}(x) + \Delta_{n+1}^{(n+1)}(x) \\
& = \Delta_0^{(n+1)}(x) f(x) \ln^{n+1}(x) + \Delta_1^{(n+1)}(x) f(x) \ln^n(x) \\
& \quad + \sum_{j=2}^n \Delta_j^{(n+1)}(x) \ln^{n+1-j}(x) + \Delta_{n+1}^{(n+1)}(x) \\
& = \sum_{j=0}^{n+1} \Delta_j^{(n+1)}(x) \ln^{n+1-j}(x)
\end{aligned}$$

$n \setminus x$	0.5	0.9	2
5	0.7057292	0.9095325	3.916667
10	0.7070978	0.9095326	4.005655
20	0.7071066	0.9095326	3.997326

FIGURE 1. Few computations of the expansion series (3.2) by sum from $k = 0$ to n for several values of n and x .

□

3. SERIES EXPANSION OF x^x

In sección 2 the derivatives of $f(x) = x^x$ has been determined such a way they only depend on the series $\Delta_k^{(n)}(x)$ and the function $f(x)$ itself. This characterization allows us to write the series expansion of x^x in the same terms:

$$(3.1) \quad f(x) = f(a) \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} \sum_{i=0}^k \Delta_i^{(k)}(a) \ln^{k-i}(a), \quad a \in \mathbb{R}$$

The particular case $a = 1$ gives a very simplified expansion series. As $\ln^{k-i}(1) = 0$ if $i < k$, every term except one are 0. Besides, $f(1) = 1$, so the result is

$$(3.2) \quad f(x) = \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} \Delta_k^{(k)}(1)$$

In subsection 4.3 we will get a more accurate expression of $\Delta_k^{(n)}(x)$. This fact will allow us to rewrite the expansion series of f .

4. PROPERTIES OF NUMBERS $\Omega(\cdot, \cdot)$

According with the definition of $\Omega(\cdot, \cdot)$ in section 1, several specific values of them can be computed easily, for instance

$$\Omega(n, 1) = \Omega(n-1, 1) + 1 = \Omega(n-2, 1) + 2 = \cdots = \Omega(1, 1) + n - 1 = n - 1$$

but it would be interesting to get an expresión allowing their computation directly. We will see below that studying sums of the numbers $\Omega(a, b)$ for any b fixed, allows us to find this expression. It will be the main result in subsección 4.1.

In 4.2 we will see the relation between numbers $\Omega(\cdot, \cdot)$ and rencontres numbers. Finally, in subsection 4.3 will be shown two examples of use for $\Omega(\cdot, \cdot)$. These examples are two families of polynomials. The first of them arises from a differential equation with the restriction of polynomials as solutions. The second family are polynomials that allow to write $\Delta_k^{(n)}(x)$ in a more accurated way.

4.1. Partial sums of numbers $\Omega(\cdot, \cdot)$.

Definition 4.1. For any fixed $i \in \mathbb{Z}_{\geq 0}$, the partial sum of n terms $\Omega(\cdot, \cdot)$ is defined by

$$(4.1) \quad S_i(n) = \sum_{j=1}^n \Omega(j, i) = \sum_{j=i+1}^n \Omega(j, i)$$

$\Omega(a, b)$	$b = 0$	$b = 1$	$b = 2$	$b = 3$	$b = 4$	$b = 5$	$b = 6$	$b = 7$	$b = 8$
$a = 1$	1	0	0	0	0	0	0	0	0
$a = 2$	1	1	0	0	0	0	0	0	0
$a = 3$	1	2	-1	0	0	0	0	0	0
$a = 4$	1	3	-3	2	0	0	0	0	0
$a = 5$	1	4	-6	8	-6	0	0	0	0
$a = 6$	1	5	-10	20	-30	24	0	0	0
$a = 7$	1	6	-15	40	-90	144	-120	0	0
$a = 8$	1	7	-21	70	-210	504	-840	720	0
$a = 9$	1	8	-28	112	-420	1344	-3360	5760	-5040

FIGURE 2. Values $\Omega(a, b)$ with $a = 1, \dots, 9$ and $b = 0, \dots, 8$

$S_0(n)$ is easy to compute due to $\Omega(a, 0) = 1$ for all $a \in \mathbb{Z}_{>1}$, and so $S_0(n) = n$. $S_1(n)$ is also easy to compute:

$$S_1(n) = \sum_{j=2}^n \Omega(j, 1) = \sum_{j=2}^n (j-1) = \sum_{j=1}^{n-1} j = \frac{(n-1)n}{2}$$

Again, these sums keep a recursive relationship:

Lemma 4.2. *Partial sums $S_i(n)$ satisfy:*

$$(4.2) \quad S_i(n) = S_i(n-1) - (i-1)S_{i-1}(n-1)$$

Proof. By the definition of $\Omega(\cdot, \cdot)$, $S_i(n)$ can be written as follows

$$\begin{aligned}
S_i(n) &= \sum_{j=i+1}^n \Omega(j, i) \\
&= \sum_{j=i+1}^n (\Omega(j-1, i) - (i-1)\Omega(j-1, i-1)) \\
&= \sum_{j=i+1}^n \Omega(j-1, i) - (i-1) \sum_{j=i+1}^n \Omega(j-1, i-1) \\
&= \sum_{j=i+2}^n \Omega(j-1, i) - (i-1) \sum_{j=i+1}^n \Omega(j-1, i-1) \\
&= \sum_{j=i+1}^{n-1} \Omega(j, i) - (i-1) \sum_{j=i}^{n-1} \Omega(j, i-1)
\end{aligned}$$

concluding

$$S_i(n) = S_i(n-1) - (i-1)S_{i-1}(n-1)$$

when applying again the definición of $S_i(n)$. □

The relationship among partials sums leads to the next lemma:

Lemma 4.3.

$$S_i(n) = (-1)^{i+1}(i-1)! \binom{n}{i+1} \text{ if } i \in \mathbb{Z}_{>0}$$

i	$S_i(n)$
0	n
1	$\frac{(n-1)n}{2}$
2	$-\frac{(n-2)(n-1)n}{6}$
3	$\frac{(n-3)(n-2)(n-1)n}{12}$
4	$-\frac{(n-4)(n-3)(n-2)(n-1)n}{20}$

FIGURE 3. $S_i(n)$ for i from 0 to 4

Proof. By induction. Let's suppose

$$S_i(n) = (-1)^{i+1}(i-1)! \binom{n}{i+1} \text{ if } i \in \mathbb{Z}_{>0}$$

(clearly right for $i = 1, \dots, 4$ as can be seen in figure 4.1). Then

$$\begin{aligned} S_{i+1}(n) &= S_{i+1}(n-1) - iS_i(n-1) \\ &= S_{i+1}(n-2) - iS_i(n-2) - iS_i(n-1) \\ &= \dots = S_{i+1}(i+1) - i \sum_{j=i+1}^{n-1} S_i(j) \end{aligned}$$

and as $S_{i+1}(i+1) = 0$,

$$\begin{aligned} S_{i+1}(n) &= -i \sum_{j=i+1}^{n-1} S_i(j) \\ &= -i \sum_{j=i+1}^{n-1} (-1)^{i+1}(i-1)! \binom{j}{i+1} \\ &= \sum_{j=i+1}^{n-1} (-1)^{i+2} i! \binom{j}{i+1} \\ &= (-1)^{i+2} i! \sum_{j=i+1}^{n-1} \binom{j}{i+1} \end{aligned}$$

but

$$(4.3) \quad \sum_{j=i+1}^{n-1} \binom{j}{i+1} = \binom{n}{i+2}$$

and therefore

$$S_{i+1}(n) = (-1)^{i+2} i! \binom{n}{i+2}$$

To finish this proof, previous equation 4.3 must be proven:

If $n = i + 2$ the equality is right. Let's suppose now that the equality is right from $i + 2$ to $n - 2$.

$$\sum_{j=i+1}^{n-1} \binom{j}{i+1} = \sum_{j=i+1}^{n-2} \binom{j}{i+1} + \binom{n-1}{i+1}$$

and by induction hypothesis,

$$\sum_{j=i+1}^{n-1} \binom{j}{i+1} = \binom{n-1}{i+2} + \binom{n-1}{i+1}$$

Finally, by Pascal's identity [3] we obtain the result. \square

As a consequence of lemma 4.3 we can get also a more accurate expression of $\Omega(a, b)$:

Proposition 4.4. For any $a, b \in \mathbb{Z}_{>0}$ with $a > b$,

$$(4.4) \quad \Omega(a, b) = (-1)^{b+1} (b-1)! \binom{a-1}{b} \text{ if } b \in \mathbb{Z}_{>0}$$

Proof. If $b = 1$,

$$(-1)^{21} 1! \binom{a-1}{1} = a-1 = \Omega(a, 1)$$

If $b > 1$,

$$\begin{aligned} \Omega(a, b) &= \Omega(a-1, b) - (b-1)\Omega(a-1, b-1) \\ &= \Omega(a-2, b) - (b-1)\Omega(a-2, b-1) - (b-1)\Omega(a-1, b-1) \\ &= \dots = \Omega(b, b) - (b-1)\Omega(b, b-1) - \dots - (b-1)\Omega(a-1, b-1) \end{aligned}$$

and as $\Omega(b, b) = 0$,

$$\Omega(a, b) = -(b-1) \sum_{j=b}^{a-1} \Omega(j, b-1) = -(b-1) S_{b-1}(a-1)$$

Finally, using lemma 4.3,

$$\begin{aligned} \Omega(a, b) &= -(b-1)(-1)^b (b-1)! \binom{a-1}{b} \\ &= (-1)^{b+1} (b-1)! \binom{a-1}{b} \end{aligned}$$

\square

Formula 4.4 gives a recursive way for the computation of $\Omega(a, 0) = 1$, but in a different way than in definition 1.1:

Corollary 4.5.

$$(4.5) \quad \Omega(a, b) = -(b-1) \sum_{j=b}^{a-1} \Omega(j, b-1) \text{ if } b \in \mathbb{Z}_{>0}$$

Proof. Just using lemma 4.3 and proposition 4.4:

$$\begin{aligned}
\Omega(a, b) &= (-1)^{b+1} (b-1)! \binom{a-1}{b} \\
&= -(b-1) (-1)^b (b-2)! \binom{a-1}{b} \\
&= -(b-1) S_{b-1}(a-1) \\
&= -(b-1) \sum_{j=b}^{a-1} \Omega(j, b-1)
\end{aligned}$$

□

4.2. Mapping between the numbers $\Omega(\cdot, \cdot)$ and rencontres numbers. Rencontres numbers count how many permutations in a set $\{1, 2, \dots, n\}$ has certain quantity of fixed points, i.e., how many derangements (partial o total). Specifically $D_{n,k}$ is the quantity of permutations of n items with k of them fixed. Rencontres numbers can be computed by

$$\begin{cases} D_{n,k} = \binom{a}{b} D_{n-k} \\ D_n = n! \sum_{i=0}^n \frac{(-1)^i}{i!} \end{cases}$$

(see [4]), which allows you to check that numbers $\Omega(\cdot, \cdot)$ match with rencontres numbers except for a multiplicative factor.

Proposition 4.6. For n and k non-negative integer numbers given with $n > k$,

$$\Omega(n, k) = \lambda_k D_{n-1, n-k-1}$$

where $\lambda_k = \left(k \sum_{i=0}^k \frac{(-1)^{i+k+1}}{i!} \right)^{-1}$.

Equivalently,

$$D_{n,k} = (n-k) \sum_{i=0}^{n-k} \frac{(-1)^{i+n-k+1}}{i!} \Omega(n+1, n-k)$$

Note: λ_k only depends on k (as the notation indicates).

Proof. Just propose equality

$$\Omega(n, k) = \lambda D_{n-1, n-k-1}$$

and find λ :

$$\begin{aligned}
\lambda &= \frac{\Omega(n, k)}{D_{n-1, n-k-1}} \\
&= \frac{(-1)^{k+1} (k-1)! \binom{n-1}{k}}{\binom{n-1}{n-k-1} k! \sum_{i=0}^k \frac{(-1)^i}{i!}} \\
&= \frac{(-1)^{k+1}}{k \sum_{i=0}^k \frac{(-1)^i}{i!}} \\
&= \left(k \sum_{i=0}^k \frac{(-1)^{i+k+1}}{i!} \right)^{-1}
\end{aligned}$$

and as λ only depends on k , we can conclude $\lambda_k = \lambda$.

□

4.3. Polynomials with coefficients $\Omega(\cdot, \cdot)$. In this section we will see two families of polynomials with numbers $\Omega(a, b)$ as coefficients. In the first example consider the family $P_n(x)$, $n \geq 1$ of monic polynomials with degree $n - 1$ satisfying the equation

$$\frac{\partial P_n}{\partial x}(x) = (n - 1)P_{n-1}(x)$$

The proposition below determines how these polynomials $P_n(x)$ are:

Proposition 4.7.

$$P_n(x) = \sum_{i=0}^{n-1} \Omega(n, i) x^{n-1-i} \quad \forall n \in \mathbb{N}$$

Proof. Let $P_n(x) = \sum_{i=0}^{n-1} a_{n,i} x^{n-1-i}$ with $a_{n,i} \in \mathbb{R}$ and $a_{n,0} = 1$ be the solution of the equation. By definition $P_1(x) = 1 = \Omega(1, 0)$. Suppose that the proposition is right up to $n - 1$. Deriving P_n one gets

$$\begin{aligned} \frac{\partial P_n(x)}{\partial x} &= \sum_{i=0}^{n-1} (n - 1 - i) a_{n,i} x^{n-2-i} \\ &= \sum_{i=0}^{n-2} (n - 1 - i) a_{n,i} x^{n-2-i} \end{aligned}$$

but by definition of P_n and induction hypothesis we have

$$\begin{aligned} \frac{\partial P_n}{\partial x}(x) &= (n - 1)P_{n-1}(x) \\ &= (n - 1) \sum_{i=0}^{n-2} \Omega(n - 1, i) x^{n-2-i} \end{aligned}$$

Equalizing both expressions we get the equations

$$(n - 1 - i) a_{n,i} = (n - 1) \Omega(n - 1, i), \quad i = 0, \dots, n - 2$$

By proposition 4.4, these equations can be written

$$\begin{aligned} a_{n,i} &= \frac{n - 1}{n - 1 - i} (-1)^{i+1} (i - 1)! \binom{n - 2}{i} \\ &= \frac{(n - 1)(n - 2)!}{(n - 1 - i)i!(n - 2 - i)!} (-1)^{i+1} (i - 1)! \binom{n - 2}{i} \\ &= \frac{(n - 1)!}{i!(n - 1 - i)!} (-1)^{i+1} (i - 1)! \binom{n - 2}{i} \\ &= (-1)^{i+1} (i - 1)! \binom{n - 1}{i} = \Omega(n, i) \end{aligned}$$

□

More in general we can write the relationship between polynomials $P_n(x)$ and their derivatives:

Proposition 4.8. for $0 < r \leq k$ given,

$$\frac{\partial^k}{\partial x^k} P_n(x) = (-1)^{r+1} r \Omega(n, r) \frac{\partial^{k-r}}{\partial x^{k-r}} P_{n-r}(x)$$

Proof. Just applying proposition 4.7 as much as needed:

$$\begin{aligned}\frac{\partial^k}{\partial x^k} P_n(x) &= (n-1) \frac{\partial^{k-1}}{\partial x^{k-1}} P_{n-1}(x) \\ &= (n-1)(n-2) \frac{\partial^{k-2}}{\partial x^{k-2}} P_{n-2}(x) \\ &= \dots = (n-1)(n-2) \dots (n-r) \frac{\partial^{k-r}}{\partial x^{k-r}} P_{n-r}(x)\end{aligned}$$

We can write it in factorial terms:

$$\frac{\partial^k}{\partial x^k} P_n(x) = \frac{(n-1)!}{(n-r-1)!} \frac{\partial^{k-r}}{\partial x^{k-r}} P_{n-r}(x)$$

which can be expressed by combinatorial numbers and use (4.4) to get the desired result:

$$\begin{aligned}\frac{\partial^k}{\partial x^k} P_n(x) &= \binom{n-1}{r} r! \frac{\partial^{k-r}}{\partial x^{k-r}} P_{n-r}(x) \\ &= (-1)^{r+1} r \Omega(n, r) \frac{\partial^{k-r}}{\partial x^{k-r}} P_{n-r}(x)\end{aligned}$$

□

Next corollaries of proposition 4.8 show specific cases of it, the specific cases $r = k$ and $r = k = n - 1$.

Corollary 4.9. For all $k \geq 0$,

$$\frac{\partial^k}{\partial x^k} P_n(x) = (-1)^{k+1} k \Omega(n, k) P_{n-k}(x)$$

Proof. As said previously, it's just proposition 4.8 when $k = r$.

□

Corollary 4.10. For all $n \geq 1$,

$$\frac{\partial^{n-1}}{\partial x^{n-1}} P_n(x) = \Gamma(n-1) = (n-1)!$$

Proof. Just use proposition 4.8 in the specific case $k = r = n - 1$ and consider $P_1(x) = 1$.

□

As second example there are polynomials definable by numbers $\Omega(\cdot, \cdot)$ which are useful in the study of $\Delta_k^{(n)}(x)$ series:

Definition 4.11.

$$(4.6) \quad Q_k(x) = x \sum_{i=0}^{k-1} \Omega(k, i) Q_{k-1-i}(x) \text{ for } k = 1, 2, \dots$$

where $Q_0(x) = \frac{1}{x}$

Once defined polynomials $Q_k(x)$, we can find a more accurate expression of $\Delta_k^{(n)}(x)$:

Proposition 4.12.

$$(4.7) \quad \Delta_k^{(n)}(x) = \binom{n}{k} \frac{Q_k(x)}{x^{k-1}}$$

Proof. For the proof of the proposition we need a lemma:

Lemma 4.13.

$$\Omega(n, i) \binom{n-1-i}{k-i} = \Omega(k+1, i) \binom{n-1}{k}$$

Proof. (of the lemma 4.13)

Just applying proposition 4.4 and operate:

$$\begin{aligned} \Omega(n, i) \binom{n-1-i}{k-i} &= (-1)^{i+1} \binom{n-1}{i} (i-1)! \binom{n-1-i}{k-i} \\ &= (-1)^{i+1} \frac{(n-1)!}{i!(n-1-i)!} (i-1)! \frac{(n-1-i)!}{(k-i)!(n-k-1)!} \\ &= (-1)^{i+1} \frac{(n-1)!(i-1)!}{i!(k-i)!(n-k-1)!} \frac{k!}{k!} \\ &= (-1)^{i+1} \binom{k}{i} \binom{n-1}{k} (i-1)! \\ &= \Omega(k+1, i) \binom{n-1}{k} \end{aligned}$$

□

According to the proposition,

$$\Delta_1^{(n)}(x) = \binom{n}{1} \frac{Q_1(x)}{x^{1-1}} = nQ_1(x)$$

but by definition

$$Q_1(x) = x\Omega(1, 0)Q_0(x) = x \frac{1}{x} = 1$$

Therefore $\Delta_1^{(1)}(x) = n \cdot 1 = n$, which is true.

For n and k fixed, suppose that $\Delta_r^{(m)}(x)$ satisfies the proposition for every r if $m < n$ and $\Delta_r^{(n)}(x)$ also for all $r \leq k$. Then,

$$\Delta_{k+1}^{(n)}(x) = \Delta_{k+1}^{(n-1)}(x) + \sum_{i=0}^k \Omega(n, i) \frac{1}{x^i} \Delta_{k-i}^{(n-1-i)}(x)$$

and applying the hypothesis we get the equation,

$$\begin{aligned} \Delta_{k+1}^{(n)}(x) &= \binom{n-1}{k+1} \frac{Q_{k+1}(x)}{x^k} + \sum_{i=0}^k \Omega(n, i) \frac{1}{x^i} \binom{n-1-i}{k-i} \frac{Q_{k-i}(x)}{x^{k-i-1}} \\ &= \binom{n-1}{k+1} \frac{Q_{k+1}(x)}{x^k} + \sum_{i=0}^k \Omega(n, i) \binom{n-1-i}{k-i} \frac{Q_{k-i}(x)}{x^{k-1}} \end{aligned}$$

but by lemma 4.13 the expression can be written

$$\begin{aligned} &\binom{n-1}{k+1} \frac{Q_{k+1}(x)}{x^k} + \sum_{i=0}^k \Omega(k+1, i) \binom{n-1}{k} \frac{xQ_{k-i}(x)}{x^k} \\ &= \binom{n-1}{k+1} \frac{Q_{k+1}(x)}{x^k} + \binom{n-1}{k} \frac{x}{x^k} \sum_{i=0}^k \Omega(k+1, i) Q_{k-i}(x) \end{aligned}$$

Now we can use the definition 4.11 to compute the sum. The equation becomes:

$$\binom{n-1}{k+1} \frac{Q_{k+1}(x)}{x^k} + \binom{n-1}{k} \frac{Q_{k+1}(x)}{x^k} = \left[\binom{n-1}{k+1} + \binom{n-1}{k} \right] \frac{Q_{k+1}(x)}{x^k}$$

Finally, by Pascal's identity [3],

$$\binom{n-1}{k+1} + \binom{n-1}{k} = \binom{n}{k+1}$$

which leads to

$$\Delta_{k+1}^{(n)}(x) = \binom{n}{k+1} \frac{Q_{k+1}(x)}{x^k}$$

□

As a consequence, the expansion series (3.1) and (3.2) in section 3 can be written as follows:

$$(4.8) \quad f(x) = f(a) \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} \sum_{i=0}^k \binom{k}{i} \frac{Q_k(a)}{a^{k-1}} \ln^{k-i}(a)$$

for any $a \in \mathbb{R}$. In the specific case $a = 1$,

$$(4.9) \quad f(x) = \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} Q_k(1)$$

REFERENCES

- [1] M. ABRAMOWITZ and I.A. STEGUN. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. 10th ed. New York, 1972.
 - [2] M. AIGNER. *Combinatorial Theory*. Erlin Heidelberg, 1979.
 - [3] T. KOSHY. *Elementary Number Theory with Applications*. USA, 2002.
 - [4] J. RIORDAN. *An Introduction to Combinatorial Analysis*. 4th ed. USA, 1967.
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